Joint work with Yaozhong Hu

Intermittency for hyperbolic Anderson equations with time-independent Gaussian noise: Stratonovich regime

Xia Chen

University of Tennessee

The 19th workshop on Markov Processes and Related Topics, Fuzhou

Our model

The model in the talk is the hyperbolic Anderson Model (HAM)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(x)u(t,x) \\ u(0,x) = 1 \;\text{ and }\; \frac{\partial u}{\partial t}(0,x) = 0 \quad x \in \mathbb{R}^d \end{cases}$$

where $\{\dot{W}(x);\ x \in \mathbb{R}^d\}$ is a mean-zero generalized stationary Gaussian field such that

$$\mathrm{Cov}\left(\dot{\mathrm{W}}(\mathrm{x}), \dot{\mathrm{W}}(\mathrm{y})\right) = \gamma(\mathrm{x}-\mathrm{y}) \quad \mathrm{x}, \mathrm{y} \in \mathbb{R}^{\mathrm{d}}$$

with $\gamma(\cdot) \geq 0$. In this talk, $\mathrm{d} = 1, 2, 3$.

Set-up of our model

Roughly speaking, our system can be viewed as the approximation of its smoothed version where $\dot{W}$ is replaced by $\dot{W}_\epsilon$.

More precisely, the hyperbolic Anderson equation is defined by following mild equation

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(dy) ds$$

where the stochastic integral on the right hand side is defined in the sense of Stratanovich, i.e., a proper limit of

$$\int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) \dot{W}_\epsilon(x) ds \quad (\text{as } \epsilon \to 0^+)$$

Mathematical set-up

and $G(t, x)$ is the fundamental solution defined by the deterministic wave equation

$$\begin{cases} \dfrac{\partial^2 G}{\partial t^2}(t, x) = \Delta G(t, x) \\ \\ G(0, x) = 0 \ \text{ and } \ \dfrac{\partial G}{\partial t}(0, x) = \delta_0(x) \quad x \in \mathbb{R}^d \end{cases}$$

Our challenge, limitation and opportunity in this talk closely related to some unique natures of $G(t, x)$, which will appear in later discussion.

Chaos expansion

Iterating the mild equation infinite times we formally have

$$u(t,x) = \sum_{n=0}^{\infty} S_n(t,x)$$

with $I_0(t,x) = 1$ and the recurrent relation

$$S_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) S_n(s,y) W(dy) ds$$

Chaos expansion

Iterating this relation we have

$$S_n(t,x) = \int_{(\mathbb{R}^d)^n} \left[\int_{[0,t]^n_<} d\mathbf{s} \left(\prod_{k=1}^{n} G(s_k - s_{k-1}, x_k - x_{k-1}) \right) \right] W(dx_1) \cdots W(dx_n)$$

where

$$[0,t]^n_< = \left\{ (s_1, \cdots, s_n) \in [0,t]^n; \quad s_1 < \cdots < s_n \right\}$$

and we adapt the conventions $x_0 = x$ and $s_0 = 0$.

Set-up of our model

Essentially, the expansion (known as Dalang-Mueller-Tribe (2008) representation)

$$u(t,x) = \sum_{n=0}^{\infty} S_n(t,x)$$

is a stochastic version of what is called Feynman-Kac formula and is formulated by Dalang, Mueller and Tribe (2008).

We recently proved that this expansion $\mathcal{L}^2$-converges, and solves the hyperbolic Anerson equation under the Dalang's condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty$$

Set-up of our model

where $\mu(\mathrm{d}\xi)$ is the spectral measure of the covariance function $\gamma(\cdot)$ determined by the relation

$$\gamma(\mathrm{x}) = \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{e}^{\mathrm{i}\xi\cdot\mathrm{x}}\mu(\mathrm{d}\xi) \qquad \mathrm{x} \in \mathbb{R}^{\mathrm{d}}$$

Prior to our progress, Balan (2022) had reached the same conclusion under a more restrictive condition

$$\int_{\mathbb{R}^{\mathrm{d}}} \left(\frac{1}{1+|\xi|^2}\right)^{1/2} \mu(\mathrm{d}\xi) < \infty$$

In this talk, our attention is on the intermittency of the system, i.e., the asymptotic behavior of the integer moments

$$\mathbb{E}\, u^p(t,x) \quad \text{or} \quad \mathbb{E}\, |u(t,x)|^p$$

as $t \to \infty$ or $p \to \infty$. In the remaining of the talk, we assume the homogeinity

$$\gamma(cx) = c^{-\alpha}\gamma(x) \qquad c > 0, \quad x \in \mathbb{R}^d$$

for some $\alpha > 0$.

In this case, Dalang's condition requests $0 < \alpha < 2$. In addtion, the fact that $\gamma(\cdot)$ is non-negative and non-negative definitive (as co-variance function) requires $\alpha \le d$. The only setting where $\alpha = d$ is allowed under the Dalang's condition is when $d = 1 = \alpha$–the setting of 1-dimensional space white noise.

Other important special cases covered by the homogeinity condition are the settings of fractional noise where

$$\gamma(x) = C_H \prod_{j=1}^{d} |x_j|^{2H_j - 2}$$

with $H_j > 1/2$ and

$$\alpha \equiv 2d - 2\sum_{j=1}^{d} H_j < 2$$

and of the Newton's potential

$$\gamma(x) = |x|^{-\alpha}$$

Main theorem

Theorem (Chen-Hu)

Assume that $0 < \alpha < 2 \wedge d$ *or that* $d = 1 = \alpha$. *Then*

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}\, u^p(t,x) = \frac{3-\alpha}{2} p^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

for any $p = 1, 2, \cdots$, *and*

$$\lim_{p\to\infty} p^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}\, |u(t,x)|^p = \frac{3-\alpha}{2} t^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

for any $t > 0$. *where*

$$\mathcal{M} = \sup_{g\in\mathcal{F}_d} \left\{ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y)\, dx dy \right)^{1/2} - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}$$

Corollary

Corollary. When $\dot{W}(x)$ $(x \in \mathbb{R})$ is an 1-dimensional white noise (i.e., $\gamma(\cdot) = \delta_0(\cdot)$), where $\alpha = d = 1$,

$$\lim_{t\to\infty} t^{-3/2} \log \mathbb{E}\, u^p(t,x) = \frac{1}{2}\sqrt[4]{\frac{3}{4}}p^{3/2} \qquad p = 1, 2, \cdots .$$

$$\lim_{p\to\infty} p^{-3/2} \log \mathbb{E}\, |u(t,x)|^p = \frac{1}{2}\sqrt[4]{\frac{3}{4}}t^{3/2} \qquad \forall t > 0$$

Remark.

In recent work by Balan, R., Chen, L. and Chen, X. (2022), the same p-limit and a slighly different t-limit

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}\,|u(t,x)|^p = \frac{3-\alpha}{2} p(p-1)^{\frac{1}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

are obtained in Skorokhod regime, under the condition $0 < \alpha < 3$.

Chaos expansion

We only prove the large-t part. First, under our intial codition $u(t,x)$ is stationary in x. So we make $x = 0$ in our proof. From

$$u(t,0) = \sum_{n=0}^{\infty} S_n(t,0)$$

we have

$$\mathbb{E}\, u^p(t,0) = \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(t,0)$$
$$= \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(t,0) = \sum_{n=0}^{\infty} t^{(4-\alpha)n} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(1,0)$$

where the last step follows from scaling.

Series decomposition of $\mathbb{E}\, u^p(t,x)$

Assume that we can prove

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\left(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^{p}S_{l_j}(1,0)\right)$$

$$=\log\left(\frac{1}{2}\right)^{3-\alpha}p^{4-\alpha}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}$$

Then the proof is completed by the computation

$$\lim_{t\to\infty}t^{-\frac{4-\alpha}{3-\alpha}}\log\sum_{n=0}^{\infty}t^{(4-\alpha)n}\left(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^{p}S_{l_j}(1,0)\right)$$

$$=\lim_{t\to\infty}t^{-\frac{4-\alpha}{3-\alpha}}\log\sum_{n=0}^{\infty}\frac{t^{(4-\alpha)n}}{(n!)^{3-\alpha}}\left(\left(\frac{1}{2}\right)^{3-\alpha}p^{4-\alpha}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}\right)^{n}$$

$$=\frac{3-\alpha}{2}p^{\frac{4-\alpha}{3-\alpha}}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

Reduction to high moment asymptotics

where the last step follows from the elementary fact that

$$\lim_{t\to\infty} t^{-1/\gamma} \log \sum_{n=0}^{\infty} \frac{\theta^n t^n}{(n!)^\gamma} = \gamma\theta^{1/\gamma} \quad (\theta, \gamma > 0)$$

with $\gamma = 3-\alpha$ and t being replaced by $t^{4-\alpha}$.

In summary, the proof of our theorem is reduced to the proof of

$$\lim_{n\to\infty} \frac{1}{n} \log (n!)^{3-\alpha} \Big(\sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(1,0) \Big)$$

$$= \log \left(\frac{1}{2}\right)^{3-\alpha} p^{4-\alpha} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}$$

Where is Feynman-Kac formula?

A similar but much more understood system is the parabolic Anderson model (PAM), where the time derivative $\partial^2 u/\partial t^2$ is replaced by $\partial u/\partial t$. For PAM, we have the Feynman-Kac representation

$$\mathbb{E}\, u^p(t,0) = \mathbb{E}_0 \exp\left\{\frac{1}{2}\sum_{j,k=1}^p \int_0^t\int_0^t \gamma\big(B(s)-B(r)\big)\,dsdr\right\}$$

or

$$\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t,0)$$

$$=\frac{1}{n!}\Big(\frac{1}{2}\Big)^n \mathbb{E}_0\Big[\sum_{j,k=1}^p \int_0^t\int_0^t \gamma\big(B(s)-B(r)\big)\,dsdr\Big]^n$$

Where is Feynman-Kac formula?

where $B_1, \cdots, B_p$ are independent Brownian motions which are independent of $\dot{W}$. Feynman-Kac reduces the problem to a problem of large deviations.

Different from PAM, the fundamental solution $G(t, x)$ does not satisfy Chapman-Kolmogorov equation and therefore is not a transition for any Markov process (such as Brownian motion). Consequently, the above Feynman-Kac representation is no-longer available in the hyperbolic setting.

However, we shall show below that the Laplacian transform changes everything we just said in the prior paragraph.

Laplacian moment representation

The following moment representation plays a fundamental role in our result:

Theorem (Representation of Stratonovich moment)

For any $\lambda > 0$, and $n = 0, 1, 2, \cdots$,

$$\int_0^\infty e^{-\lambda t} S_n(t,0)dt$$
$$= \frac{1}{n!}\frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n \int_0^\infty \exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_0\Big[\int_0^t \dot{W}(B(s))ds\Big]^n dt \quad a.s.$$

where $B(s)$ is a d-dimensional Brownian motion independent of $\dot{W}$ with $B(0) = 0$, and "$\mathbb{E}_0$" is the expectation with respect to the Brownian motion.

Mathematical set-up

This relation largely depends on the specific form

$$\widehat{G}(t,\xi) \equiv \int_{\mathbb{R}^d} G(t,x)e^{i\xi\cdot x}dx = \frac{\sin(|\xi|t)}{|\xi|}.$$

of the Fourier transform of the fundamental solution $G(t,x)$. Unlike its Fourier transform, $G(t,x)$ takes very different forms in different dimensions. In the dimensions $d = 1, 2, 3$, for example,

$$G(t,x) = \begin{cases} \frac{1}{2}1_{\{|x|\le t\}} & d = 1 \\ \frac{1}{2\pi}\frac{1_{\{|x|\le t\}}}{\sqrt{t^2-|x|^2}} & d = 2 \\ \frac{1}{4\pi t}\sigma_t(dx) & d = 3 \end{cases}$$

Mathematical set-up

where $\sigma_t(dx)$ is the surface measure on $\{x \in \mathbb{R}^3;\ |x| = t\}$.

The reason that we limit our discussion to $d = 1, 2, 3$ because these are only cases where $G(t, x) \geq 0$.

A crucial and elementary observation is

$$\int_0^\infty e^{-\lambda t} G(t, x) dt = \frac{1}{2} \int_0^\infty e^{-\lambda^2 t/2} p(t, x) dt \qquad x \in \mathbb{R}^d$$

for any $\lambda > 0$, where $p(t, x)$ is the density of $B(t)$:

$$p(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{ -\frac{|x|^2}{2t} \right\} \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

Laplacian moment representation

Indeed, the both sides has the same Fourier transform

$$
\begin{aligned}
&\int_{\mathbb{R}^d} e^{i\xi\cdot x}\left[\int_0^\infty e^{-\lambda t}G(t,x)dt\right]dx\\
&=\int_0^\infty e^{-\lambda t}\frac{\sin|\xi|t}{|\xi|}dt=\frac{1}{\lambda^2+|\xi|^2}\\
&=\frac{1}{2}\int_0^\infty e^{-\lambda^2 t/2}\exp\left\{-\frac{1}{2}|\xi|^2 t\right\}dt\\
&=\int_{\mathbb{R}^d} e^{i\xi\cdot x}\left[\frac{1}{2}\int_0^\infty e^{-\lambda^2 t/2}p(t,x)dt\right]dx
\end{aligned}
$$

for every $\xi\in\mathbb{R}^d$.

Laplacian moment representation

Recall the elementary identity

$$\lambda \int_0^\infty e^{-\lambda t} \int_{[0,t]_<^n} ds_1 \cdots ds_n \prod_{k=1}^n \varphi_k(s_k - s_{k-1})$$
$$= \prod_{k=1}^n \int_0^\infty e^{-\lambda t} \varphi_k(t) dt$$

Here we recall the notation

$$[0, t]_<^n = \Big\{(s_1, \cdots, s_n) \in [0, t]^n; \ \ s_1 < \cdots < s_n\Big\}$$

and follow the convention $s_0 = 0$.

Laplacian moment representation

Therefore,

$$\int_0^\infty e^{-\lambda t} S_n(t,0)dt$$

$$= \int_0^\infty dt e^{-\lambda t} \int_{(\mathbb{R}^d)^n} d\mathbf{x} \int_{[0,t]^n_<} d\mathbf{s} \left(\prod_{k=1}^n G(s_k - s_{k-1}, x_k - x_{k-1}) \right)$$

$$\times \left(\prod_{k=1}^n \dot{W}(x_k) \right)$$

$$= \lambda^{-1} \int_{(\mathbb{R}^d)^n} d\mathbf{x} \left(\prod_{k=1}^n \int_0^\infty e^{-\lambda t} G(t, x_k - x_{k-1}) dt \right) \left(\prod_{k=1}^n \dot{W}(x_k) \right)$$

Laplacian moment representation

$$= \lambda^{-1}\Big(\frac{1}{2}\Big)^n \int_{(\mathbb{R}^d)^n} d\mathbf{x}\Big(\prod_{k=1}^n \int_0^\infty e^{-\lambda^2 t/2} p(t, x_k - x_{k-1})dt\Big)$$

$$\times \Big(\prod_{k=1}^n \dot{W}(x_k)\Big)$$

$$= \frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n \int_0^\infty dt \exp\Big\{-\frac{\lambda^2}{2}t\Big\} \int_{[0,t]^n_<} d\mathbf{s}$$

$$\times \int_{(\mathbb{R}^d)^n} d\mathbf{x}\Big(\prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1})\Big)\Big(\prod_{k=1}^n \dot{W}(x_k)\Big)$$

Laplacian moment representation

Given $(s_1, \cdots, s_n) \in [0,t]^n_<$, the random vector $\big(B(s_1), \cdots, B(s_n)\big)$ has the joint density

$$f_{s_1,\cdots,s_n}(x_1, \cdots, x_n) \triangleq \prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1})$$

So we have (recall that $x_0 = 0$)

$$\int_{(\mathbb{R}^d)^n} d\mathbf{x} \left(\prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1}) \right) \left(\prod_{k=1}^n \dot{W}(x_k) \right)$$
$$= \mathbb{E}_0 \prod_{k=1}^n \dot{W}\big(B(s_k)\big)$$

Laplacian moment representation

Finally,

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t} S_n(t,0)dt \\
&= \frac{\lambda}{2}\left(\frac{1}{2}\right)^n \int_0^\infty dt \exp\left\{-\frac{\lambda^2}{2}t\right\} \int_{[0,t]^n_<} d\mathbf{s}\mathbb{E}_0 \prod_{k=1}^n \dot{W}\big(B(s_k)\big) \\
&= \frac{1}{n!}\frac{\lambda}{2}\left(\frac{1}{2}\right)^n \int_0^\infty \exp\left\{-\frac{\lambda^2}{2}t\right\}\mathbb{E}_0\left[\int_0^t \dot{W}\big(B(s)\big)ds\right]^n dt
\end{aligned}
$$

$\square$

Laplacian moment representation

Corollary (Laplacian moment representation)

Given $\lambda_1, \cdots, \lambda_p > 0$,

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p \lambda_j t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^p S_{l_j}(t_j, 0)$$

$$= \Big(\frac{1}{2}\Big)^{3n} \frac{1}{n!} \Big(\prod_{j=1}^p \frac{\lambda_j}{2}\Big) \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \lambda_j^2 t_j\Big\}$$

$$\times \mathbb{E}_0\Big[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s) - B_k(r)\big)\, ds dr\Big]^n \quad n = 0, 1, 2, \cdots$$

where $B_1(t), \cdots, B_p(t)$ *are independent* d*-dimensional Brownian motions starting at 0.*

Laplacian moment representation

Proof.

$$\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p \lambda_j t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p S_{l_j}(t_j,0)$$

$$= \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p \int_0^\infty e^{-\lambda_j t} S_{l_j}(t,0)dt$$

$$= \sum_{l_1+\cdots+l_p=2n} \Big(\prod_{j=1}^p \frac{\lambda_j}{2}\Big(\frac{1}{2}\Big)^{l_j}\Big) \prod_{j=1}^p \frac{1}{l_j!} \int_0^\infty dt e^{-\lambda^2 t/2} \mathbb{E}_0\Big[\int_0^t \dot{W}(B(s))ds\Big]^{l_j}$$

$$= \Big(\frac{1}{2}\Big)^{2n}\Big(\prod_{j=1}^p \frac{\lambda_j}{2}\Big) \int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \lambda_j^2 t_j\Big\}$$

$$\times \frac{1}{(2n)!}\mathbb{E}_0\Big[\sum_{j=1}^p \int_0^{t_j} \dot{W}(B_j(s))ds\Big]^{2n}$$

Laplacian moment representation

The remaining of the proof relies on the fact that conditioning on the Brownian motions,

$$\sum_{j=1}^{p}\int_0^{t_j}\dot{W}\big(B_j(s)\big)ds$$

is normal with zero mean and the variance

$$\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr$$

Laplacian moment representation

Consequently,

$$\mathbb{E}\left[\sum_{j=1}^{p}\int_0^{t_j}\dot{W}\big(B_j(s)\big)ds\right]^{2n}$$

$$=\frac{(2n)!}{2^n n!}\left[\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\right]^{n}$$

Together with the computation by far, this completes the proof. $\square$

Laplacian moment asymptotics

We now start the proof of the main theorem. The first step is to show

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$
$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}$$

Taking $\lambda_1=\cdots=\lambda_p=1$ in Laplacian moment representation, it is equivalent to

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\,\mathbb{E}_0\bigg[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)\,dsdr\bigg]^n=\log 2^4\mathcal{M}^{\frac{4-\alpha}{2}}$$

Upper bound of Laplacian moment

By Parseval's indentity

$$\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr=\int_{\mathbb{R}^d}\mu(d\xi)\left|\sum_{j=1}^{p}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right|^2$$

$$=(t_1+\cdots+t_p)^2\int_{\mathbb{R}^d}\mu(d\xi)\left|\sum_{j=1}^{p}\frac{t_j}{t_1+\cdots+t_p}\frac{1}{t_j}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right|^2$$

$$\le(t_1+\cdots+t_p)\sum_{j=1}^{p}t_j\int_{\mathbb{R}^d}\mu(d\xi)\left|\frac{1}{t_j}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right|^2$$

$$=(t_1+\cdots+t_p)\sum_{j=1}^{p}\frac{1}{t_j}\int_0^{t_j}\int_0^{t_j}\gamma\big(B_j(s)-B_j(r)\big)dsdr$$

$$\overset{d}{=}(t_1+\cdots+t_p)\sum_{j=1}^{p}t_j^{\frac{2-\alpha}{2}}\int_0^{1}\int_0^{1}\gamma\big(B_j(s)-B_j(r)\big)dsdr$$

Upper bound of Laplacian moment

where the inequality follows from Jensen and the last step from Brownian scaling and homogenity of $\gamma(\cdot)$.

So we have

$$\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n$$

$$\le (t_1+\cdots+t_p)^n\mathbb{E}_0\Big[\sum_{j=1}^p t_j^{\frac{2-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B_j(s)-B_j(r)\big)dsdr\Big]^n$$

$$=(t_1+\cdots+t_p)^n\sum_{l_1+\cdots+l_p=n}\frac{n!}{l_1!\cdots l_p!}$$

$$\times\prod_{j=1}^p t_j^{\frac{2-\alpha}{2}l_j}\mathbb{E}_0\Big[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\Big]^{l_j}$$

Upper bound of Laplacian moment

and therefore

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\left\{-\frac{1}{2}\sum_{j=1}^p t_j\right\}\mathbb{E}_0\left[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s)-B_k(r)\big)\,dsdr\right]^n$$

$$\le n! \sum_{l_1+\cdots+l_p=n} \frac{1}{l_1!\cdots l_p!}\left\{\prod_{j=1}^p \mathbb{E}_0\left[\int_0^1\int_0^1 \gamma\big(B(s)-B(r)\big)\,dsdr\right]^{l_j}\right\}$$

$$\times \int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p (t_1+\cdots+t_p)^n \exp\left\{-\frac{1}{2}\sum_{j=1}^p t_j\right\}\prod_{j=1}^p t_j^{\frac{2-\alpha}{2}l_j}$$

$$= n! \sum_{l_1+\cdots+l_p=n} \frac{1}{l_1!\cdots l_p!}\left\{\prod_{j=1}^p \mathbb{E}_0\left[\int_0^1\int_0^1 \gamma\big(B(s)-B(r)\big)\,dsdr\right]^{l_j}\right\}$$

$$\times 2^p 2^{\frac{4-\alpha}{2}n}\left(\prod_{j=1}^p \Gamma\left(1+\frac{2-\alpha}{2}l_j\right)\right)\Gamma\left(p+\frac{2-\alpha}{2}n\right)^{-1}\Gamma\left(p+\frac{4-\alpha}{2}n\right)$$

Upper bound of Laplacian moment

From the large deviation for self-intersection local time

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{-\alpha/2}\mathbb{E}_0\left[\int_0^1\int_0^1\gamma(B_s-B_r)dsdr\right]^n=\log 2^\alpha\left(\frac{4\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2}}$$

That means: We are allowed to do the replacement

$$\mathbb{E}_0\left[\int_0^1\int_0^1\gamma(B(s)-B(r))dsdr\right]^{l_j}\approx(l_j!)^{\alpha/2}\left(2^\alpha\left(\frac{4\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2}}\right)^{l_j}$$

in our computation

Upper bound of Laplacian moment

Using Stirling formula

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\left\{-\frac{1}{2}\sum_{j=1}^p t_j\right\}\mathbb{E}_0\left[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s)-B_k(r)\big)\,dsdr\right]^n$$

$$\preceq n! 2^{\frac{4-\alpha}{2}n}\left(\frac{2-\alpha}{2}\right)^n\left(2^\alpha\left(\frac{4\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2}}\right)^n\Gamma\left(p+\frac{2-\alpha}{2}n\right)^{-1}$$

$$\times\,\Gamma\left(p+\frac{4-\alpha}{2}n\right)\sum_{l_1+\cdots+l_p=n} 1$$

$$\approx (n!)^2 2^{4n}\mathcal{M}^{\frac{4-\alpha}{2}n}\begin{pmatrix} n+p-1 \\ p-1\end{pmatrix} \approx (n!)^2 2^{4n}\mathcal{M}^{\frac{4-\alpha}{2}n}$$

Upper bound of Laplacian moment

In summary, we have established the upper bound

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\,\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\le\log 2^4\mathcal{M}^{\frac{4-\alpha}{2}}$$

In the following we prove the lower bound

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\,\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\ge\log 2^4\mathcal{M}^{\frac{4-\alpha}{2}}$$

Lower bound of Laplacian moment

By Cauchy-Schwartz inequality

$$\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr=\int_{\mathbb{R}^d}\mu(d\xi)\bigg|\sum_{j=1}^{p}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\bigg|^2$$

$$\geq\left[\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\bigg(\sum_{j=1}^{p}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\bigg)\right]^2$$

$$=\left[\sum_{j=1}^{p}\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\bigg(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\bigg)\right]^2$$

for any non-negative $f(\xi)$ with $f(-\xi)=f(\xi)$ and

$$\int_{\mathbb{R}^d}|f(\xi)|^2\mu(d\xi)=1$$

Lower bound of Laplacian moment

Therefore

$$\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n$$

$$\geq \mathbb{E}_0\Big[\sum_{j=1}^p\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big)\Big]^{2n}$$

$$=\sum_{l_1+\cdots+l_p=2n}\frac{(2n)!}{l_1!\cdots l_p!}\prod_{j=1}^p\mathbb{E}_0\Big[\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\Big)\Big]^{l_j}$$

$$=(2n)!\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^p\int_{(\mathbb{R}^d)^{l_j}}\mu(d\xi)\Big(\prod_{k=1}^{l_j}f(\xi_k)\Big)$$

$$\times\int_{[0,t_j]^{l_j}_<}d\mathbf{s}\prod_{k=1}^{l_j}\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big\}$$

Lower bound of Laplacian moment

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}\mathbb{E}_0\Big[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s)-B_k(r)\big)\,ds\,dr\Big]^n$$

$$\geq (2n)! \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p \int_{(\mathbb{R}^d)^{l_j}} \mu(d\xi)\Big(\prod_{k=1}^{l_j} f(\xi_k)\Big)$$

$$\times \int_0^\infty dt e^{-t/2} \int_{[0,t]^{l_j}_<} d\mathbf{s} \prod_{k=1}^{l_j} \exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big\}$$

$$= 2^{2n+1}(2n)! \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p \int_{(\mathbb{R}^d)^{l_j}} \mu(d\xi) \prod_{k=1}^{l_j} f(\xi_k)\Big(1+\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big)^{-1}$$

Lower bound of Laplacian moment

The spectral method yields that

$$\lim_{n\to\infty}\frac{1}{n}\log\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n f(\xi_k)\Big(1+\Big|\sum_{i=k}^n\xi_i\Big|^2\Big)^{-1}$$

$$=\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\Big]\triangleq\rho(f)$$

Consequently, we are allowed to do the replacement

$$\int_{(\mathbb{R}^d)^{l_j}}\mu(d\xi)\prod_{k=1}^{l_j}f(\xi_k)\Big(1+\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big)^{-1}\approx\rho(f)^{l_j}$$

Lower bound of Laplacian moment

Therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\,\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\ge\log 2^4\rho^2(f)$$

Finally, the desired lower bound follows from the relation

$$\sup_{\|f\|_2=1}\rho^2(f)=\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)\Big[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\Big]^2$$
$$=\mathcal{M}^{\frac{4-\alpha}{2}}$$

Taking "Laplacian inverse"

In summary, we have reached the conclusion that

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$
$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}$$

As the last step, we now prove that

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\Big(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big)$$
$$=\log\Big(\frac{1}{2}\Big)^{3-\alpha}p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}$$

Taking “Laplacian inverse”

We first prove the upper bound

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\geq \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\Big(\min_{1\leq j\leq p} t_j,\ 0\Big)$$

$$=\Big\{\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big\}$$

$$\times \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\leq j\leq p} t_j\Big)^{(4-\alpha)n}$$

Taking “Laplacian inverse”

By the fact for the i.i.d. $\exp(1)$-random variables $\tau_1, \cdots, \tau_p$, $\min\limits_{1\le j\le p}\tau_j \sim \exp(p)$,

$$\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\le j\le p} t_j\Big)^{(4-\alpha)n}$$
$$= p\int_0^\infty e^{-pt}t^{(4-\alpha)n}dt = \Big(\frac{1}{p}\Big)^{(4-\alpha)n}\Gamma\Big(1+(4-\alpha)n\Big)$$

Using Stirling formula we obtain the desired upper bound

$$\limsup_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\Bigg(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Bigg)$$
$$\le \log\Big(\frac{1}{2}\Big)^{3-\alpha}p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}$$

Taking "Laplacian inverse"

The same argument can be adapted for the lower bound with some extra effort. Let $\delta > 0$ be fixed but small. When $(t_1, \cdots, t_p) \in \left(n^{\frac{4-\alpha-\delta}{p}}, n^{\frac{4-\alpha+\delta}{p}}\right)^p$,

$$t_j \le \frac{4-\alpha+\delta}{p} n \le \frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k \quad j = 1, \cdots, p$$

So we have

$$\sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(t_j, 0) \le \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}\left(\frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k, 0\right)$$

$$= \left(\frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k\right)^{(4-\alpha)n} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(1, 0)$$

Taking "Laplacian inverse"

Therefore,

$$\int_{\left(n^{\frac{4-\alpha-\delta}{p}},\, n^{\frac{4-\alpha+\delta}{p}}\right)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\le \Big\{\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big\}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta}\Big)^{(4-\alpha)n}$$

$$\times\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\le j\le p} t_j\Big)^{(4-\alpha)n}$$

$$=\Big\{\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big\}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta}\Big)^{(4-\alpha)n}$$

$$\times\Big(\frac{1}{p}\Big)^{(4-\alpha)n}\Gamma\Big(1+(4-\alpha)n\Big)$$

Concentration of exponential times

To complete the proof for the lower bound, therefore, all we need is to show

$$\int_{\left(\frac{n(4-\alpha-\delta)}{p},\frac{n(4-\alpha+\delta)}{p}\right)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\sim \int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0) \qquad (n\to\infty)$$

for any small $\delta>0$. This will be proved later.

Concentration of exponential times

First recall that

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}$$

Working harder on the moment representation, we can prove that for any $\lambda_1,\cdots,\lambda_p>0$

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p \lambda_j t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\le \log 2\mathcal{M}^{\frac{4-\alpha}{2}}+\frac{4-\alpha}{2}\sum_{j=1}^p\frac{\lambda_j^{-2}\log\lambda_j^{-2}}{\lambda_1^{-2}+\cdots+\lambda_p^{-2}}$$

The correspondent lower bound is very likely, but we are not able to prove it.

Concentration of exponential times

Define the probability measures $\mu_n(A)$ on $(\mathbb{R}^+)^p$

$$\mu_n(A) = \frac{\displaystyle\int_A dt_1 \cdots dt_p \exp\Big\{ -\sum_{j=1}^p t_j \Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^p S_{l_j}(t_j, 0)}{\displaystyle\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{ -\sum_{j=1}^p t_j \Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^p S_{l_j}(t_j, 0)}$$

For our purpose, we need to show that

$$\lim_{n\to\infty} \mu_n\Big(\big(\frac{n(4-\alpha-\delta)}{p}, \ \frac{n(4-\alpha+\delta)}{p}\big)^p \Big) = 1$$

For any $(\theta_1, \cdots, \theta_p) \in \mathbb{R}^p$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_{(\mathbb{R}^+)^p} \exp\Big\{ \sum_{j=1}^p \theta_j t_j \Big\} \mu_n(dt_1 \cdots dt_p) \le \Lambda(\theta_1, \cdots, \theta_p)$$

Concentration of exponential times

where

$$\Lambda(\theta_1, \cdots, \theta_p) = \frac{4-\alpha}{2}\sum_{j=1}^p \frac{(1-\theta_j)^{-2}\log(1-\theta_j)^{-2}}{(1-\theta_1)^{-2}+\cdots+(1-\theta_p)^{-2}}$$

when $\theta_1, \cdots, \theta_p < 1$, and $\Lambda(\theta_1, \cdots, \theta_p) = +\infty$ if otherwise.

By the upper bound of Gärtner-Ellis theorem,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(nF) \le -\inf_{(t_1,\cdots,t_p)\in F}\Lambda^*(t_1, \cdots, t_p)$$

for every close set $F\subset(\mathbb{R}^+)^p$, where

$$\Lambda^*(t_1, \cdots, t_p) = \sup_{\theta_1,\cdots,\theta_p}\left\{\sum_{j=1}^p \theta_j t_j - \Lambda(\theta_1, \cdots, \theta_p)\right\}$$

Concentration of exponential times

It is not easy (perhaps) and unnecessary to find the close form of $\Lambda^*(\cdot)$. Clearly, $\Lambda^*(t_1, \cdots, t_p) \geq 0$. Further, we claim that $\Lambda^*(t_1, \cdots, t_p) > 0$ whenever $t_j \neq \frac{4-\alpha}{p}$ for any $1 \leq j \leq p$.

Indeed, assume $(t_1, \cdots, t_p) \in (\mathbb{R}^+)^p$ that makes $\Lambda^*(t_1, \cdots, t_p) = 0$. We must have

$$\sum_{j=1}^p \theta_j t_j \leq \frac{4-\alpha}{2} \sum_{j=1}^p \frac{(1-\theta_j)^{-2} \log(1-\theta_j)^{-2}}{(1-\theta_1)^{-2} + \cdots + (1-\theta_p)^{-2}}$$

for every $(\theta_1, \cdots \theta_p) \in (-\infty, 1)^p$. In particular, for given j, take $\theta_j = \theta$ and $\theta_k = 0$ for $k \neq j$:

$$\theta t_j \leq \frac{4-\alpha}{2} \frac{(1-\theta)^{-2} \log(1-\theta)^{-2}}{(p-1) + (1-\theta)^{-2}}$$

Concentration of exponential times

Thus,

$$t_j \le (4-\alpha)\frac{(1-\theta)^{-2}}{(p-1)+(1-\theta)^{-2}}\frac{1}{\theta}\log(1-\theta)^{-1} \qquad \theta > 0$$

$$t_j \ge (4-\alpha)\frac{(1-\theta)^{-2}}{(p-1)+(1-\theta)^{-2}}\frac{1}{\theta}\log(1-\theta)^{-1} \qquad \theta < 0$$

Letting $\theta \to 0^+$ and $\theta \to 0^-$ separately, we have $t_j = \frac{4-\alpha}{p}$.

Write $G_\delta = (\frac{4-\alpha-\delta}{p}, \ \frac{4-\alpha+\delta}{p})^p$. We have, therefore,

$$\inf_{(t_1,\cdots,t_p)\in G_\delta^c} \Lambda^*(t_1,\cdots,t_p) > 0$$

Taking $F = G_\delta^c$ in the large deviation upper bound,

$$\limsup_{n\to\infty} \frac{1}{n}\log \mu_n(nG_\delta^c) < 0$$

Concentration of exponential times

In particular,

$$\int_{nG_\delta} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$
$$\sim \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0) \quad (n\to\infty)$$

That is what we try to prove. $\square$

Future challenges

Evidence suggests that compared to the parabolic Anderson models, hyperbolic Anderson equation should be more toleratent to the singularity of the Gaussian noise. We therefore **conjecture** that the renormalized (in the sense of Stratanovich) equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + (\dot{W}(x) - \infty) u(t,x) \\[2em] u(0,x) = 1 \ \text{ and } \ \dfrac{\partial u}{\partial t}(0,x) = 0 \quad x \in \mathbb{R}^d \end{cases}$$

Future challenges

has solution in $\mathcal{L}^2(\Omega)$ (and therefore in all positive moment) under the assumption

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^3}\mu(d\xi) < \infty$$

Further, we believe that under the homogeneinity

$$\gamma(cx) = c^{-\alpha}\gamma(x) \quad c > 0, \;\; x \in \mathbb{R}^d$$

the same result of intermittency holds. This is particularly true for the setting of two-dimensional space white noise.

All of these are concluded positively in the Skorodhod regime.

Future challenges

Another case breaking our argument is when the Gaussian noise depends on time, i.e., a mean zero Gaussian field $W(t,x)$ with the covariance

$$Cov\left(\dot{W}(t,x),\dot{W}(s,y)\right)=\gamma_0(t-s)\gamma(x-y)$$

where $\gamma_0(\cdot)=\delta_0(\cdot)$ or $|\cdot|^{-\alpha_0}$ (with $0<\alpha_0<1$).

In the case when $\gamma_0(\cdot)=\delta_0(\cdot)$, Balan and Song (2019) establish the existence under the Dalang's condition and compute the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\,u^2(t,x)$$

Future challenges

In the Skorodhod regime, the existence is established (Chen-Deya-Song-Tindal (2024+)) under the assumption

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\frac{3-\alpha_0}{2}} \mu(d\xi) < \infty$$

References

Balan, R. M., Chen, L. and Chen, X. Exact asymptotics of the stochastic wave equation with time-independent noise. *Annales de l'Institut Henry Poincare* **58** (2022)1590-1620.

Balan, R. M. . Stranovich solution for the wave equation. *J. Theor. Probab.* **35** (2022) 2643-2689.

Balan, R. and Song, J. Second order Lyapunov exponents for parabolic and hyperbolic Anderson models. *Bernoulli* **25** (2019) 3069-3089.

References

Chen, X., Deya, A. Song, J. and Tindel, S. Solving the hyperbolic model I: Skorohod setting. *Annales de l'Institut Henri Poincare* (accepted)

Dalang, R. C.; Mueller, C. Tribe, R. A. Feynman-Kac-type formula for the deterministic and stochastic wave equations and other P.D.E.'s. *Trans. Amer. Math. Soc.* **360** (2008) 4681-4703.

Thank you!